PERSPECTIVE APPEARANCES and REPRESENTATIONS

Understanding perspective in direct vision theory.
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1. Introduction

The laws of perspective are rooted in geometrical optics and not, as widely assumed, in rules and conventions of “realistic looking” painting, nor in the art of delineating 3D objects onto a 2D planar canvas, paper or screen.

This is an inquiry into direct vision theory and its relevance to perspective. It looks into the mathematical and physical laws attributed to their cause and effect and that purport to explain their nature.

What physical laws and facts actually define and determine perspective? Are these real physical laws, or just theories or only conventions? Are they demonstrable or hypothetical? Are the laws actually correct? Are they confirmed in reality by common sense? Are the applications of any such laws, fully understood and correctly practiced?

We investigate two theorems and their consequences, essential to understanding perspective and the difference between APPEARANCES and REPRESENTATIONS.

They are derived from a geometrical optics model and not from an artistic or representative model. We use basic rational geometry to demonstrate proportions and determine the veracity of certain propositions and theorems.

Some of these propositions are intuitive but contain subtleties. Others are counter intuitive and occulted but shown to be true.

In accepted thinking, the interpreted results of such determinations are often condensed into triggers, to seed an intended thinking or to solidify a “known fact”. Simple memes such as “double the distance means half the size” or other similar catch phrases are easily parroted. Such truisms are to be accepted as self evident. These clichés are not to be doubted.

In truth they should be recognized as conditioned responses that capture only select parts of the truth.

What is the true nature of perspective? Let’s see what the eye sees.

2. Quick Summary

- Direct vision via the naked eye, is conveyed in straight lines between an object and the eye via visual rays (by some unknown means), when object and eye are both assumed to be in the same medium.
- The eye is globular in form and the retina is a curved surface. Visual rays entering the eye, converge and cross each other in the eye, where they then diverge in equal angles, falling on the retina in the back part of the eye.
- The optic angle determines APPEARANCE. This is the angle subtended at the eye. It is a 2 dimensional angle in a 3 dimensional space, that an object subtends at a point.

Theorem 1: Objects appear to have that proportion to each other, respectively, as the angles under which they are seen.

- Near objects in the line of sight, can appear to touch each other or be of equal size. We cannot judge from sight alone; which object is farther away or which is larger.
- Two or more UNEQUAL objects at an EQUAL distance from the eye and seen direct DO NOT appear exactly of the proportion of their respective magnitudes.
- Equal parts of the same object on the same side of that point where it is cut by the axis of the eye, DO NOT appear equal.
- Objects of equal magnitude, seen at different distances DO NOT appear exactly in the ratio of their several distances.
- Distances must be, to all sense, infinite, before they can appear in equal ratio to their distances. The visual rays must be parallel or nearly so.

Theorem 2: Parallel straight lines, however situated, when extended, appear to approach towards each other. If produced indefinitely ("infinitely"), they will appear to meet in a point at an indefinite ("infinite") distance.

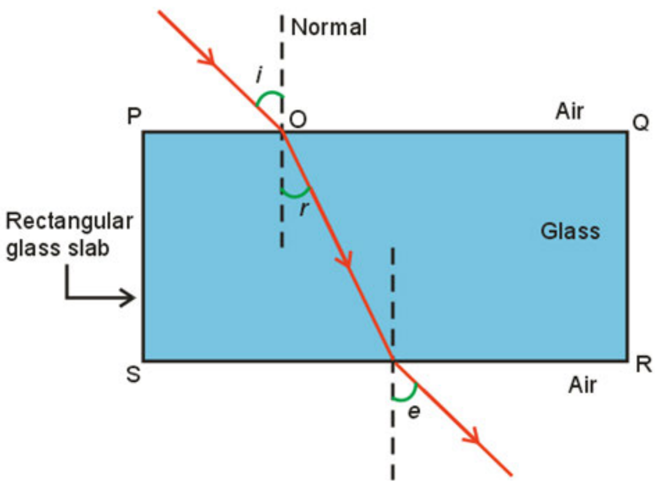
- A horizontal line, plane or continued level surface (e.g. water), below the eye, as is the will gradually appear to ascend to the eye line. Similarly, a line or plane above the eye will appear to descend to the eye line.
- Every line/plane in which the eye is NOT situate, will appear to approach towards, and at length appear to meet another line/plane which passes through the eye and is parallel to it.
- When representing objects on a plane we can NEVER maintain the same proportions as their true appearance when the objects are in different positions that are distanced or removed farther or closer from the eye line.
- Objects of various dimensions can appear to be equal. This is NOT DUE to their distances, but due to their positions with respect to the eye.
- True representations and true appearance of objects, can ONLY be correctly depictured on figures and surfaces where visual rays intersect perpendicular to the surface, such as a circle or a sphere.

3. Assumptions and definitions

3.1. Direct vision via the naked eye, is conveyed in straight lines between an object and the eye via visual rays, when object and eye are both assumed to be in the same medium.

a) Visual rays are the straight lines from all parts of an object to the eye, under which, or by means of which, the object is seen, or supposed to be seen. These visual rays are modelled as straight lines connecting points (to the eye), and are basic geometric elements in optic theory.

b) Refraction of light happens at the boundaries of different mediums (e.g. air and glass or water). Light travels in straight lines in the mediums before, during and after directional changes.



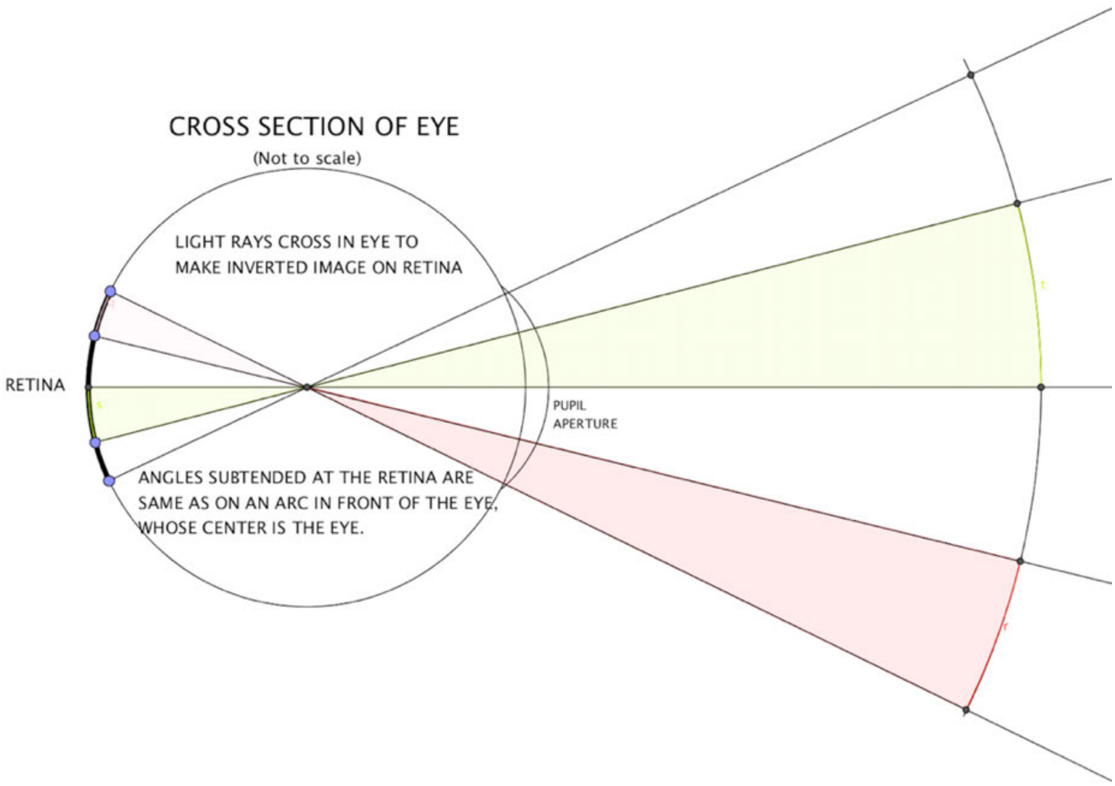
c) Any light lacking luminescence or constrained by refractive angles to remain in only one medium, can never reach the eye, so can play no part in our vision. The "last mile" to the eye is straight and direct.

3.2. The eye is globular in form and the retina is a curved surface.

a) Should this be doubted, or it be argued that the eye continuously changes its internal shape such that the Retina is always a flat plane, then google "eyeball" and count the flickers of your eyes per second, to recognize that this would be nonsensical.

b) We are considering only one eye (monocular vision), but there are no reasonable grounds to contend, that with both eyes (binocular vision), the geometry will change. In fact, with binocular vision, two circles will appear as one. To see this put a toilet roll to each eye, then move them together till the ends meet.

3.3. Visual rays entering the eye, converge and cross each other in the eye, where they then diverge in equal angles, falling on the retina in the back part of the eye. This is the same principle as a pinhole camera or camera obscura.



a) These angles made on the retina, are the same as those on any concentric arc circle, with a radius such, that it can be drawn <u>between</u> the <u>eye</u> and the <u>object</u>. We can <u>always</u> draw such a circle or an arc segment of a circle, <u>outside</u> the eye.

b) All visual rays intersect the retina and any concentric arc, <u>perpendicular</u> to the <u>perimeter</u> or boundary surface due to their equally curved nature.

c) The <u>axis</u> of the eye (eye line) is a <u>straight line</u> through the centre of the eye and the centre of the aperture or pupil. When the pupil is direct, the point where the axis meets the retina, is the centre of the retina and distinct vision.

d) Whenever we say "appear" or "appearance" or "apparent", we mean the <u>angle(s) subtended at the eye</u> by the visual rays connecting the eye with the object(s).

3.4. The diagrams are cross sectional views of a 3D space. We are viewing a 2D cross sectional representation of an eye that is viewing an object. We are determining the visual rays connecting the eye with the extremes of the object(s).

a) The representation is a top view or plan view, a view from above. The same representation is also a side on view, or profile view, a view from the side. The views are equal because the eye is spherical and so we can slice it horizontally and look

down from above at some instantaneous scene, or slice it vertically and then look from the side.

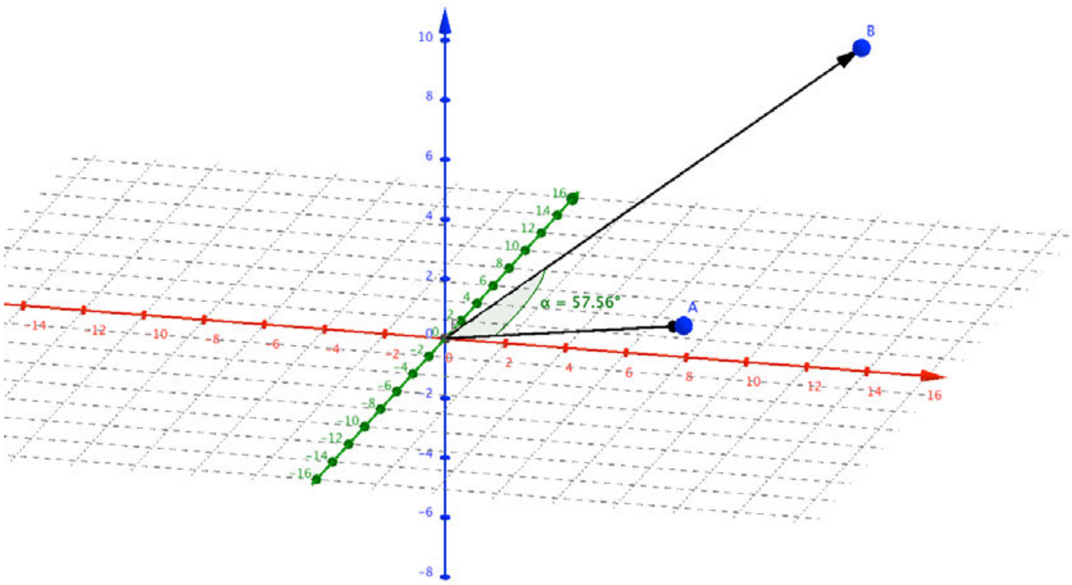
b) In the side view we are viewing the up/down axis (Y Axis) and the forward/backward axes (Z Axis). In the top view, we are viewing the left/right axis (X Axis) and the forward/backward axes (Z Axis). The forward/backward direction is depicted to the right of the eye in the diagrams.

c) These are extrinsic views and we are seeing what the eye cannot see, namely the depth (or forward direction). In the intrinsic view (from the point of view of the eye), there is no true distance or size information. There are only depth cues from color, object overlapping, light and shadow, etc. The eye's intrinsic view of figures and shapes is 2 dimensional only, up/down (Y Axis) and left/right (X Axis).

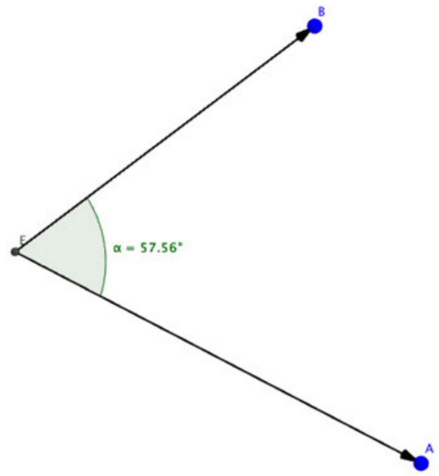
3.5. The Optic Angle determines APPEARANCE. This is the angle subtended at the eye. It is a 2 dimensional angle in a 3 dimensional space, that an object subtends at a point.

a) It is a measure of how large the object appears to an observer at a specific point. It can be thought of as either a plane or a solid angle. Both are 2 dimensional angles, with the exception that one is supposed to be on a plane 2D surface (planar), whilst the other has a direction in 3D space (solid).

Note: The eye is situated at the origin. The Axis are drawn only to assist initial understanding.



b) Each line from the eye (origin) to the object A or B is dependent only on two things. The situation of the eye and the object itself. The angle measured between these two lines is a relation between the two points A and B. It is independent of any pre defined "world axes" or Cartesian planes. If the eye is in a different position, then the APPEARANCES (angular relation between the two objects, subtended at the eye) are different. This difference is less, the farther the objects are from the eye.



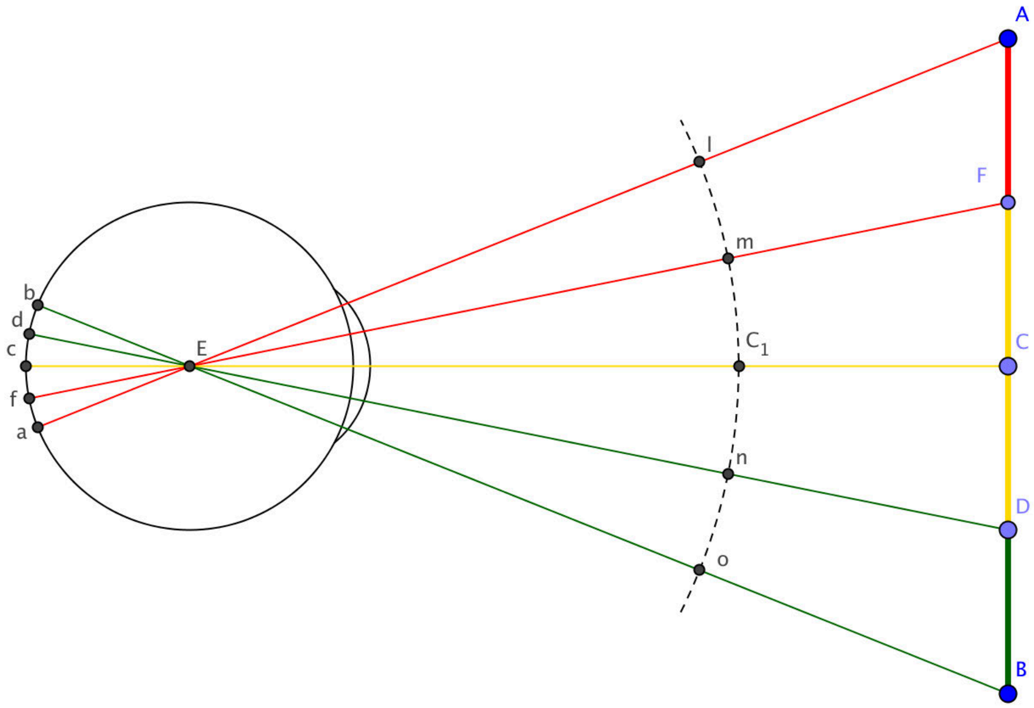
c) If the object AB has only length and no other dimension, it can be considered a straight line. The visual rays from the eye, E to both extremes (both ends) form a plane angle AEB, called the optic angle under which the line AB is seen. This is a geometric model. We don't actually see lines going from the eye to the objects or even lines joining objects. We see only the separation of the two objects, a one directional visual extension called length or distance between them.

4. Perspective of direct vision

4.1. Theorem 1

Objects appear to have that proportion to each other, respectively, as the angles under which they are seen.

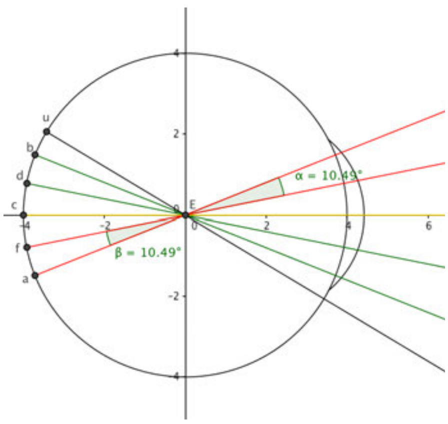
Let afcdb represent a section of the retina, part of a concave sphere. Let AB be an object directly before the eye and divided in any ratio by the points D and F.



Proof:

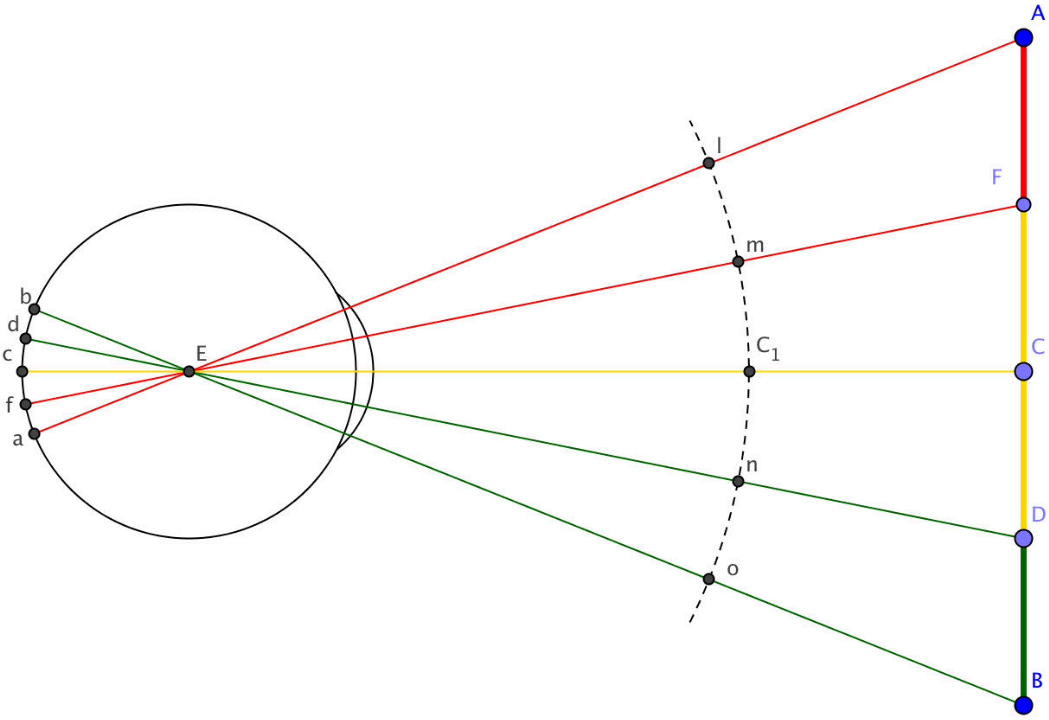
The object AB is seen under the angle AEB by means of the visual rays EA and EB. The parts of AB, such as AF, FD and DB are seen under the respective angles AEF, FED and DEB.

The opposite angles aEf, fEd and dEb, are respectively equal to the angles AEF, FED and DEB, as shown by the example below of AEF and aEf.



The images af, fd and db on the retina of the original parts AF, FD and DB are the measures of those angles respectively. The retina is a portion of a sphere so that each arc segment is equally distant from the centre E.

The arc af is to the angle aEf, as the arc fd is to the angle fEd, and as db is to dEb.



So the images or pictures on the retina and consequently the apparent magnitudes of objects are in the <u>same ratio or proportion</u> <u>to each other</u>, as the <u>angles they subtend at the eye</u>.

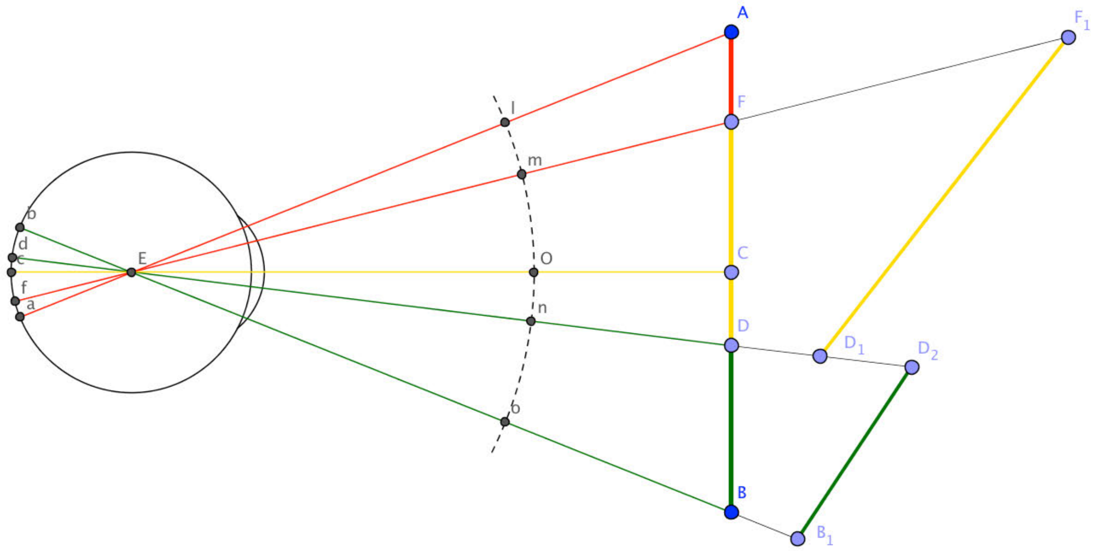
The arc lmC_1no, of any radius between the eye and the object intercepts the visual rays. The portions of this arc (lm, mn and no) measure the angles AEF, FED and DEB, respectively.

Therefore, the parts AF, FD and DB of the object AB, will appear to the eye at E, <u>in proportion to the arc segments</u> lm, mn and no.

QED.

4.2. Corollary 1.1

Near objects in the line of sight, can appear to <u>touch</u> each other <u>or be of equal size</u>. We <u>cannot</u> judge from sight alone, which object is <u>farther</u> away or which object is <u>larger</u>.



<u>Proof</u>

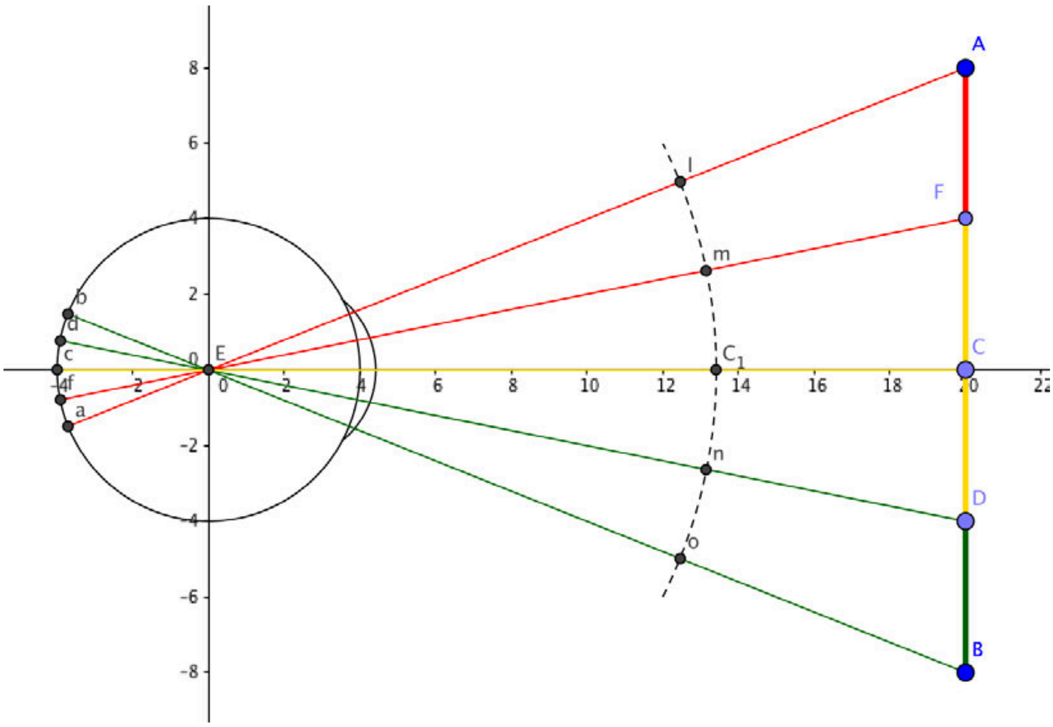
Suppose the Visual Rays EF, ED and EB are extended to F_1, D_1, D_2 and B_1. Then any object such as F_1D_1 and D_2B_1, in the same direction as FD and DB and touching the same visual rays (EF, ED and EB) will coincide with FD and DB. The lesser object FD (or DB) will hide the greater one F_1D_1 (or B_1D_2) from sight and they will appear to be of equal magnitude.

QED.

4.3. Corollary 1.2

Two or more <u>UNEQUAL</u> objects at an <u>EQUAL distance from the eye</u> and seen direct <u>DO NOT always appear</u> exactly of the <u>proportion</u> of their respective <u>magnitudes</u>.

Let AB be an object, perpendicular to the eye line EC and <u>bisected</u> in the point C. Let AC and CB, also be <u>bisected</u> in the points F and D.



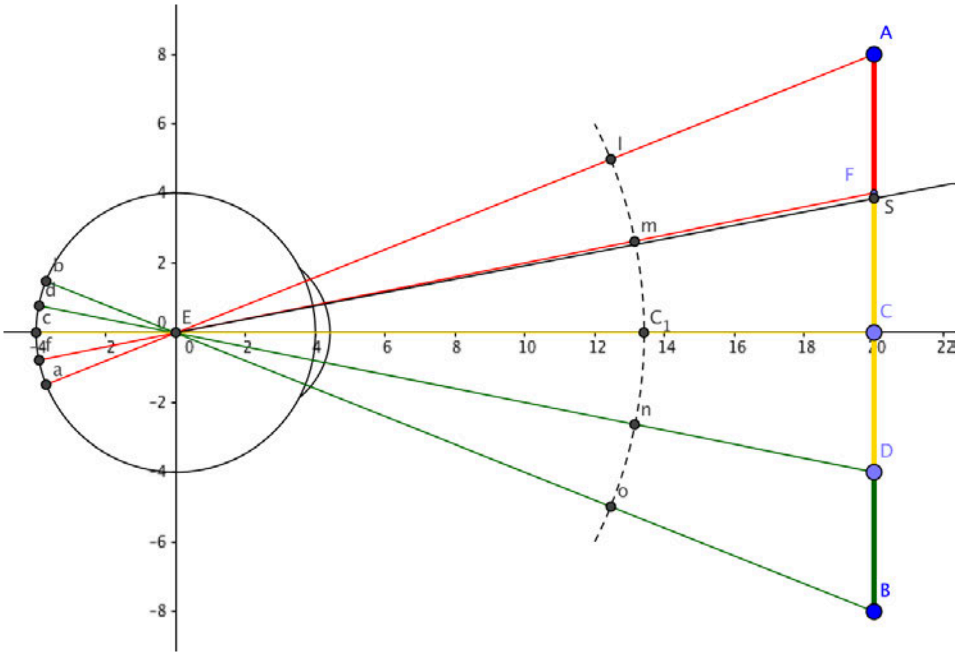
So AF, FC, CD and DB are all of <u>equal</u> length, and AB is <u>double</u> FD.

However, they <u>DO NOT</u> <u>appear</u> of that <u>proportion</u> (i.e. double the angle) when seen from E.

<u>Proof:</u>

AF is <u>equal</u> to FC, but the angles AEF and FEC under which they are seen, are <u>unequal</u>.

If the angle AEC was bisected, then the straight line ES bisecting it, will cut AC in the point S, in the ratio of EA to EC.



Also, EA is greater than EC since EC is perpendicular to AC and a hypotenuse is always larger. So AS is greater than SC and the angles AES, SEC under which <u>they</u> are seen are <u>equal</u>.

That means that the <u>equal</u> parts AF, FC are seen under <u>unequal</u> angles and must therefore <u>appear unequal</u> (by Theorem 1).

Also, because the angle CEF is greater than FEA, the equal part CF will appear greater than FA. We can repeat the same steps to show that the other equal part CD also appears greater than DB (this is not shown in the diagram).

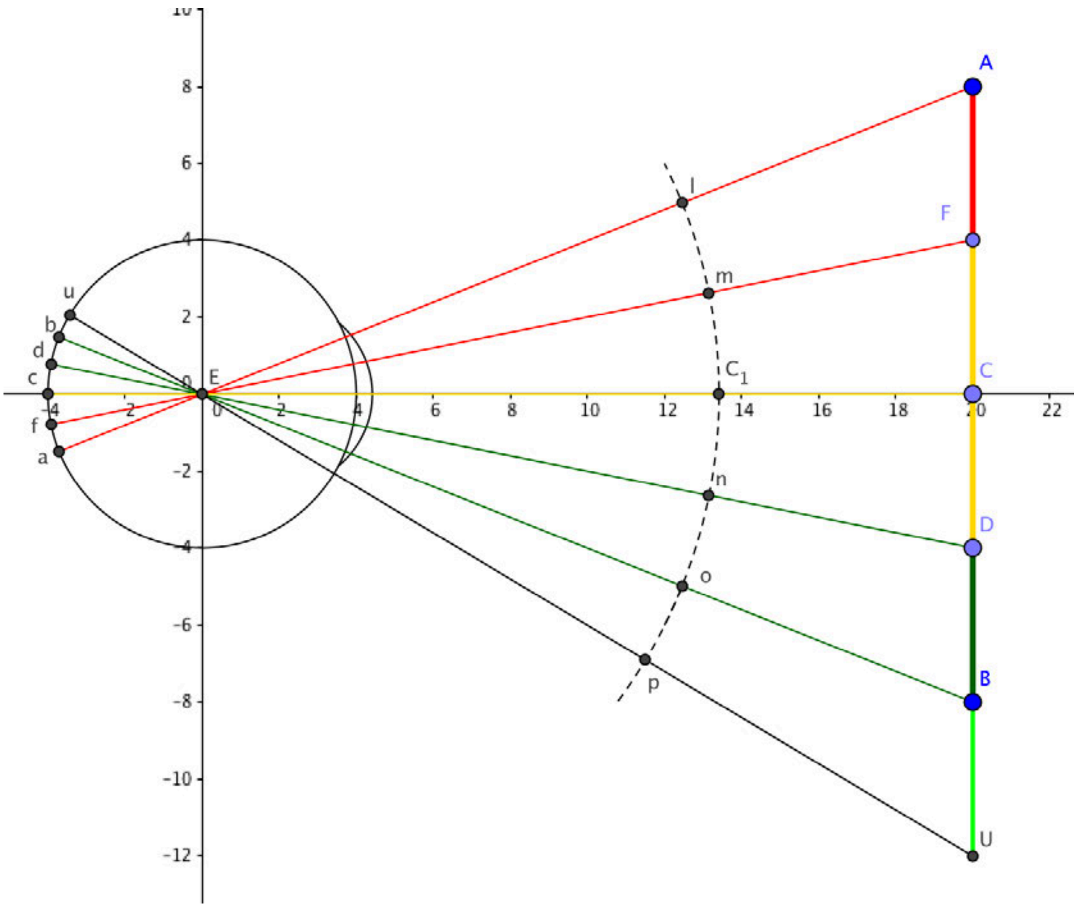
So, CD + CF together make FD, and this <u>appears</u> <u>greater</u> than half of AB, <u>although</u> they are <u>really of the same size</u>.

QED.

4.4. Corollary 1.3

<u>Equal</u> parts of the an object on the <u>same side</u> of that <u>point</u> where it is <u>cut</u> by the <u>axis</u> of the eye, <u>DO NOT</u> appear equal.

If AB is extended and additional equal divisions are created such as BU (light green), then these parts will <u>appear</u> continually <u>less</u> because they are seen under <u>less</u> <u>angles</u>.



The angle UEB is less than BED which is itself less than DEC. This can be seen by the measures po, on and nC_1 of those angles as the arc segments of the circle.

Or in other words, the equal part <u>nearer</u> to the eye will <u>appear</u> (i.e. the angle it subtends at the eye) <u>larger</u> than the part that is further away. Equal parts that are <u>further</u> and <u>further</u> away from the eye, will <u>appear</u> at an angle that is <u>less</u> and <u>less</u>.

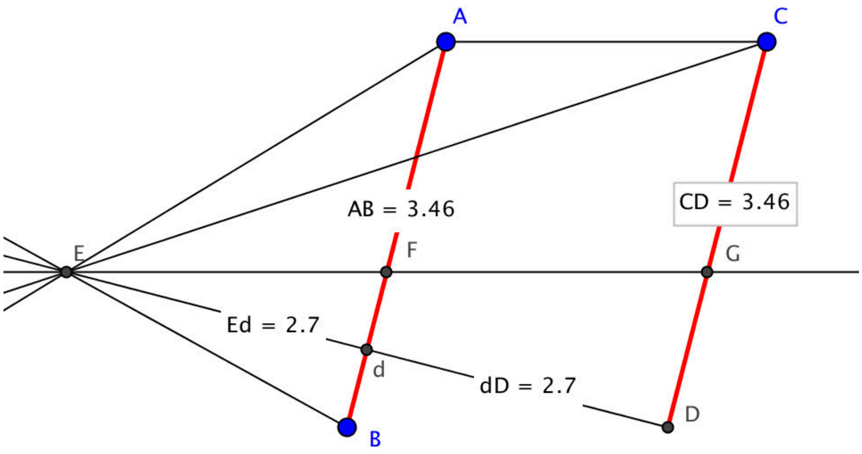
Note that, the farther away any two or more objects are from the eye, the nearer they will <u>appear in the proportion</u> of <u>their</u> <u>respective</u> magnitudes. This is because the tangents of small angles deviate less from the ratio of angles, than the tangents of larger angles.

4.5. Corollary 1.4

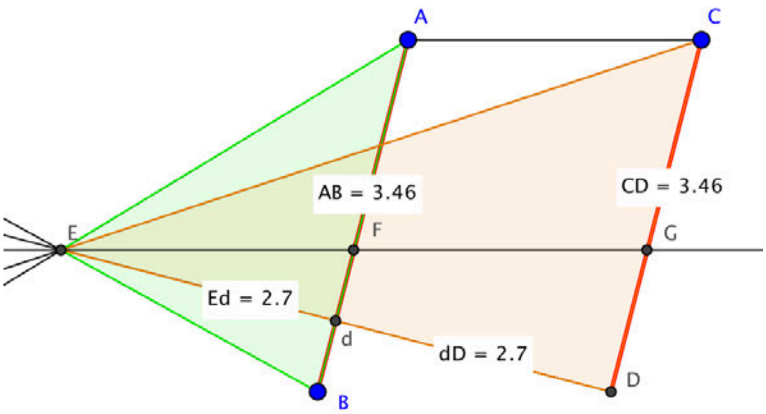
Objects of <u>equal</u> magnitude, seen at <u>different</u> distances <u>DO NOT</u> <u>appear exactly</u> in the <u>ratio</u> of their distances.

Let AB be an object at any distance (in this case Ed) from the eye situated at E. Let CD be another object, parallel to AB and of equal length. Let CD be twice the distance from the eye than AB is, so that ED is double Ed.

Note: In this diagram we are no longer showing the retina, only the centre E where the rays would meet and diverge to the retina.

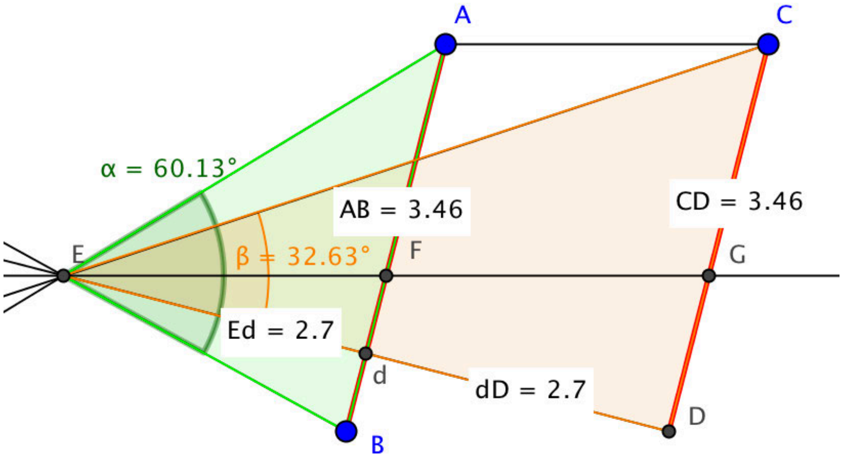


The visual rays EA, EB, EC and ED are drawn connecting the object extremes to the eye. The objects are then seen under the angles AEB (green) and CED (orange).



The angle AEB (green) is NOT the double of the Angle CED (orange), although AB, CD are of equal length and CD is double the distance of AB from the eye (ED = Ed + dD =2Ed)

This time, before showing a detailed proof, we will use the software to tell us what the angles are.



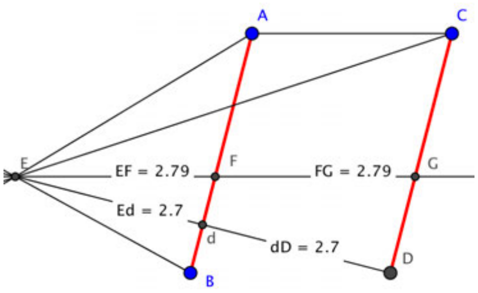
AEB (green) = 60.13 degrees and CED (orange) = 32.63 degrees.

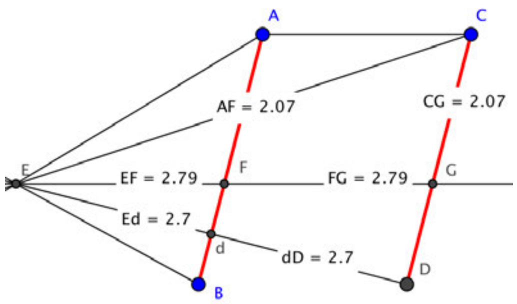
So AEB is NOT double of CED.
CD (equal to AB) appears greater than half of AB, though seen at twice its distance.

QED.

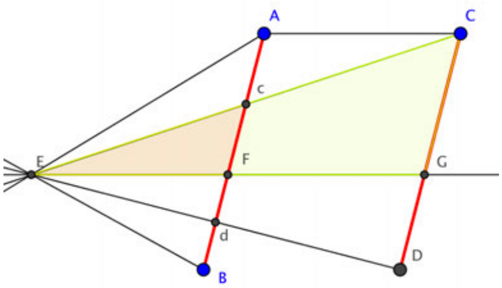
Detailed proof:

AB is parallel to CD, so EF : FG :: Ed : dD and EF : EG :: Ed : ED (from Euclids Elements).
But Ed is equal to half of ED (our hypothesis), so that EF is equal to FG.





Since FG is parallel to AC and AF to CG, then AF = CG.
Since AF is parallel to CG the triangles EcF and ECG are similar, so that EF : EG :: Fc : GC.



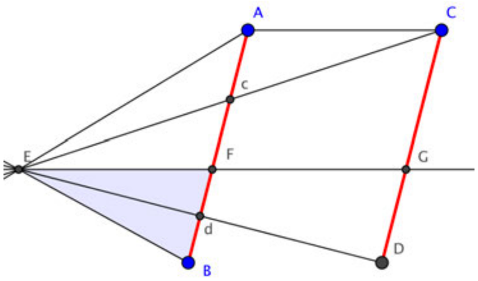
But EF is equal to half EG, so Fc is equal to half GC. Therefore, AF (equal to CG) is bisected in c, so that Ac must be equal to cF.

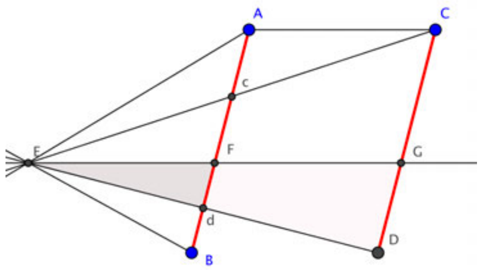
Now since Ac is equal to cF, the angle AEc is not equal to cEF. If the angle AEF were bisected by EC, then AF would be cut in the ratio of EA to EF. In other words, Ac would be to cF as EA is to EF (Ac : cF :: EA : EF).

Now the angle EFA is obtuse, so EA must be longer than E and it has already been proven that objects appear to the eye in proportion to the angles under which they are seen. Consequently, Ac does not <u>appear</u> equal to cF, because Ac = cF.

CG being seen under the same angle (CEG) as cF, appears to the eye at E, to be equal to cF. Then CG, equal to AF, appears greater than half AF, given that ED is the double of Ed. Similarly, the remaining part GD, equal to FB, can be shown to appear equal to half FB.

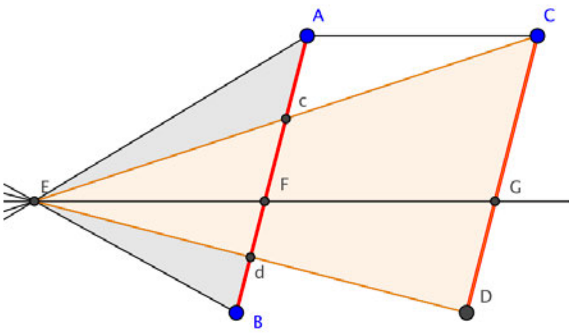
For let the Angle FEB be bisected by ED, cutting BF and DG perpendicularly, so the triangle BEF is isosceles. Then EB = EF. Consequently, BD = dF and they are seen under equal angles, Bed = dEF.





Therefore, DG being seen under the same angle (DEG) as dF (dEF), will appear equal to dF, which is half of BF.

But if equals are added to “unequals”, the sums are unequal. So, the angle CEG + GED, which is equal to CED is in total greater than the sum AEc + dEB.

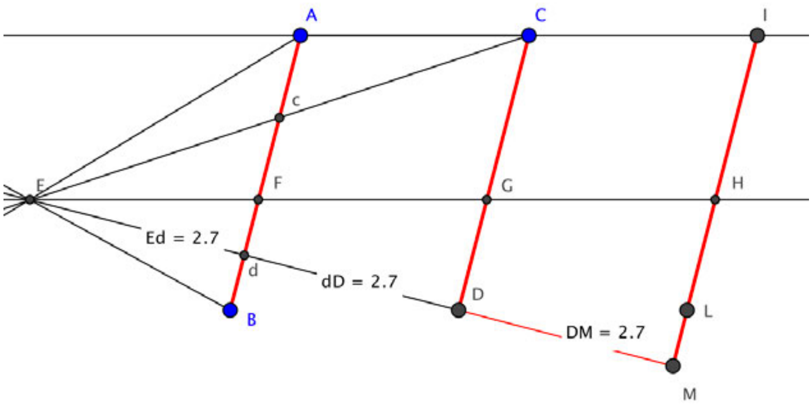


Consequently, CD (in reality equal to AB) <u>appears</u> <u>greater</u> than <u>half</u> of AB, <u>though</u> seen at <u>twice</u> its distance.

QED.

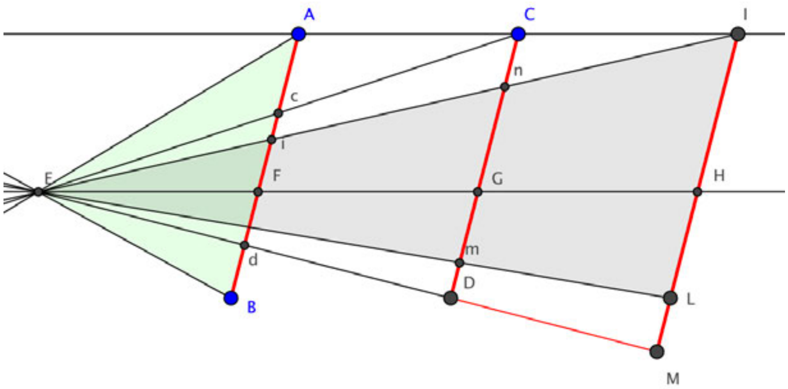
4.6. More examples

Let IL be another object, <u>equal and parallel</u> to AB, and at <u>three times the distance</u>, then IL will <u>appear</u> <u>greater</u> <u>than one third</u> of AB.



IL is extended to M and the perpendicular ED is also extended to M. Then EM is the distance of IL from the eye. We have also extended AC to I and EG to H.

It can be proven similar to the previous example, that the angle AEF is <u>NOT trisected</u> by the straight line EI and that the angle IEL is <u>more than one third</u> of AEB.



It can also be proven that if we let the remaining part HL (equal to FB) be equal to two thirds of HM, that it will also NOT APPEAR in that proportion to the eye at E.

<u>To summarize:</u>

From the above examples it is hopefully clear that objects <u>DO NOT APPEAR</u> in that <u>proportion to each other, as their respective distances</u>.

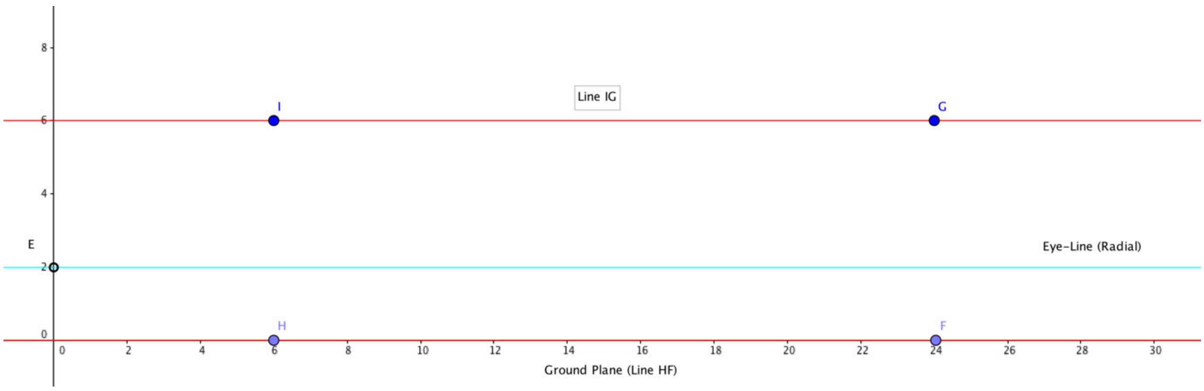
Objects of different magnitudes, at equal distances, do not appear in the proportion of their respective magnitudes. Neither do objects of equal magnitude, seen at different distances appear exactly in the ratio of their several distances.

As the distance of an object is increased, the angle under which it is seen, decreases in a reciprocal proportion. <u>Distances</u> must be, to all sense, <u>infinite</u>, <u>before</u> they can <u>appear</u> in <u>equal</u> <u>ratio</u> to their distances. The visual rays must be parallel or nearly so.

4.7. Theorem 2

Parallel straight lines, however situated, when extended, appear to approach towards each other. If produced indefinitely ("infinitely"), they will appear to meet in a point at an indefinite ("infinite") distance.

Let's observe a person (and their eye) situated between two horizontal parallel planes. The planes extend into and out of the paper/screen/canvas. From the side, we see these horizontal planes as lines. They are seen "edge on" in the geometric diagram. We are depicting a side view of vision being performed by this eye.

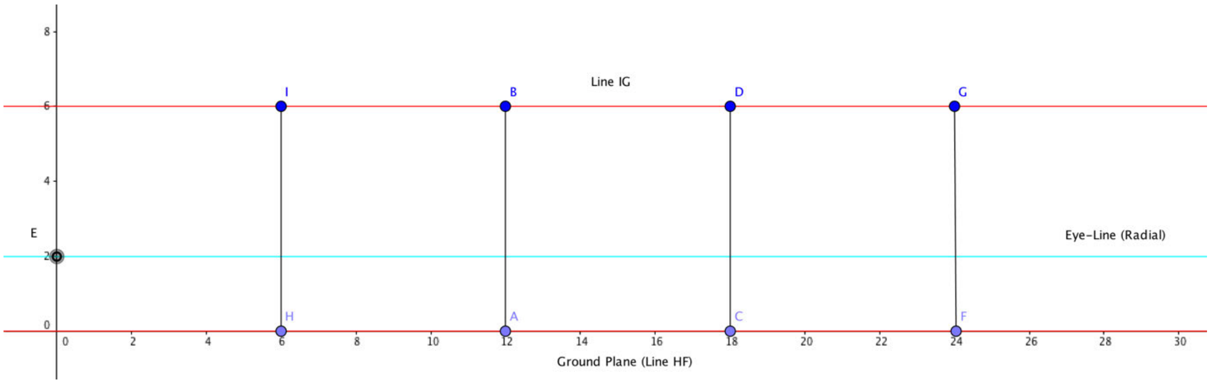


IG and HF are two (red) horizontal parallel lines. E is the position of the eye. The cyan line ER is the eye line. The eye line is a straight line from the eye, parallel to the ground plane. The eye is in a natural position looking directly forward in the direction of nose and knees.

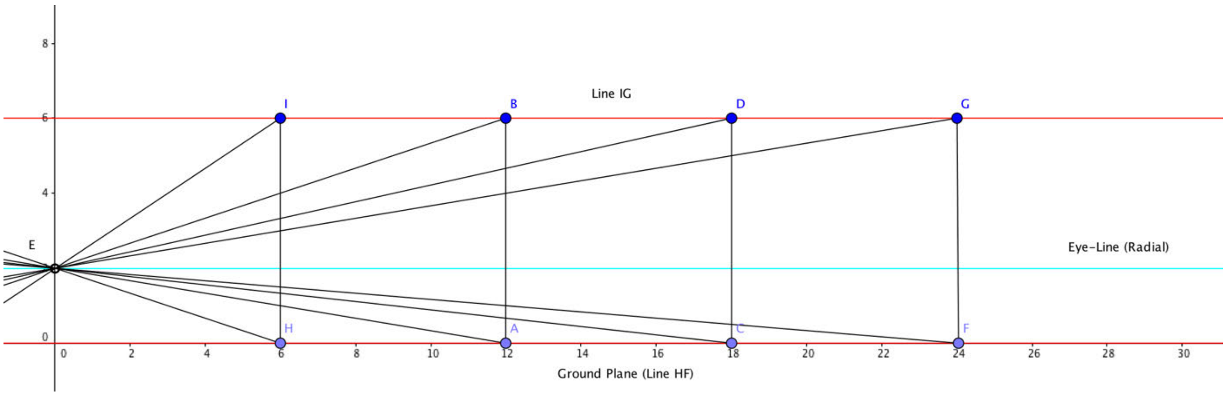
The eye at E, can be situated anywhere between the two lines (or planes). In this example, E is closer to the bottom line (2 units removed) and farther from the top line (4 units removed). This can be thought of as a 2m large person standing on the ground with a ceiling or roof that is 6m high.

Note: Alternatively, we can interpret the diagram as a top view and think of it as a person standing between parallel railway tracks that are 6m apart. The planes are then vertical, and perpendicular to the ground, so that we still see them "edge on". The eye is situated closer to one of the railway tracks. The track on its right hand side (2m) is closer than the track on its left hand side (4m). The subjective (intrinsic) "right and left", side to side, clockwise and anti clockwise, about turn a circle, etc., is from the point of view of the eye. The objective (extrinsic) view is now "top and bottom", higher and lower, up and down, etc.

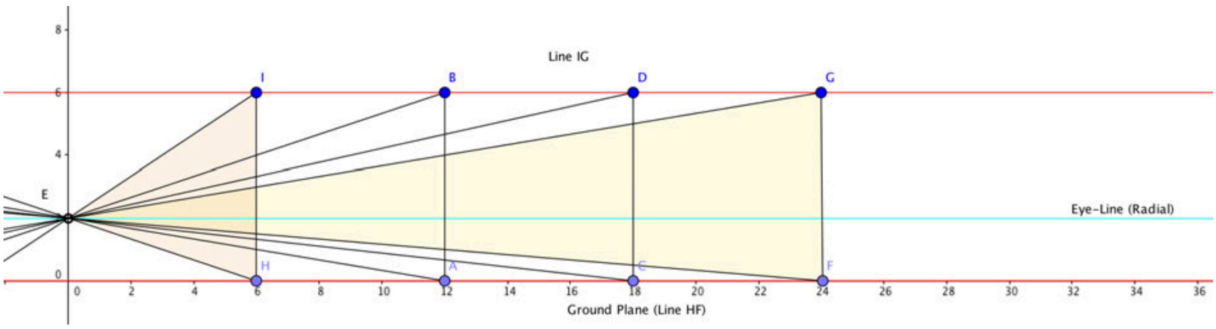
Now we add identical perpendicular lines HI, AB, CD and FG, equally spaced (6 units) between the lines. These perpendicular lines show that the two red lines IG and HF remain parallel throughout.



Now, visual rays from the object extremes (HI, AB, CD, FG) are drawn in straight lines to the eyes.

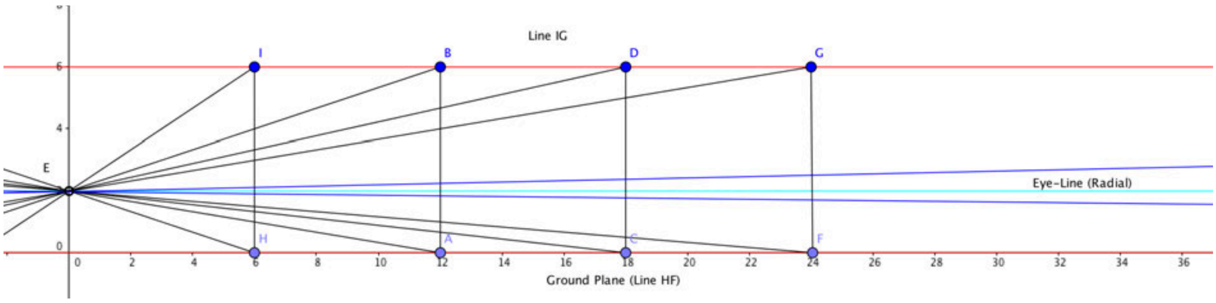


It is obvious that ΔFEG < ΔCED < ΔAEB < ΔHEI. The space between the lines IG and HF is seen with these angles and that these angles decrease with increased distance from the eye. In the diagram below this is seen clearly by the colouring of ΔFEG and ΔHEI.



So the space between the top and bottom horizontal lines, IG and HF diminishes, becoming more and more difficult to discern. At a great distance, the angles will be so small, as to be insensible. The interval between the lines will then be lost to sight. The space between them, and consequently any objects in between, are no longer distinct and are said to vanish.

The eye line is the parallel line to which the lines IG and HF appear to approach. If all the lines are produced <u>indefinitely</u>, then they will <u>appear</u> to meet in the same point in the distance.

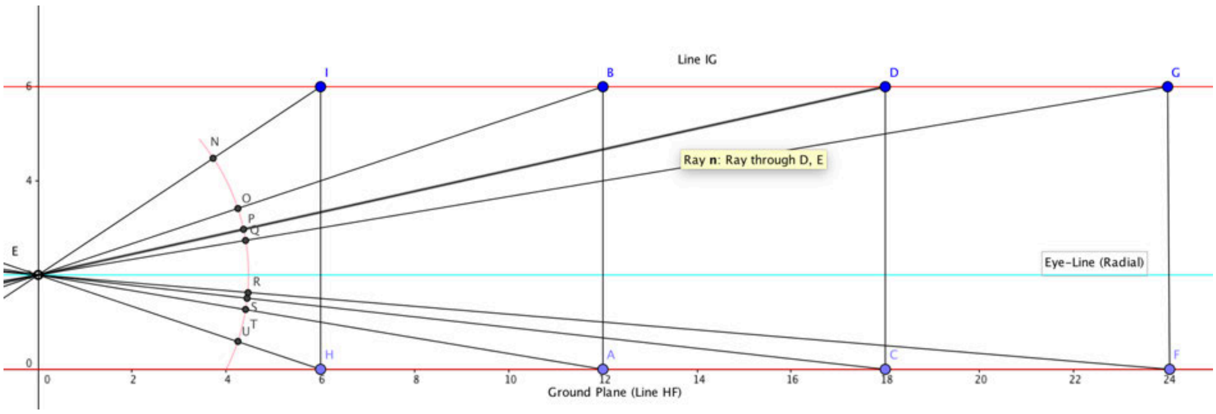


The two blue lines are examples of visual rays, from the extremes of an object far to the right, and these lines approach ever closer to the eye line. At a distance corresponding to the limits of the optical instrument, these two blue lines and the eye line will coincide and seem to be the same line.

ER, the eye line, may be nearer to one line than to the other, as in this example, where E is closer to HF than it is to IG, in the proportion 2:4 or 1:2.

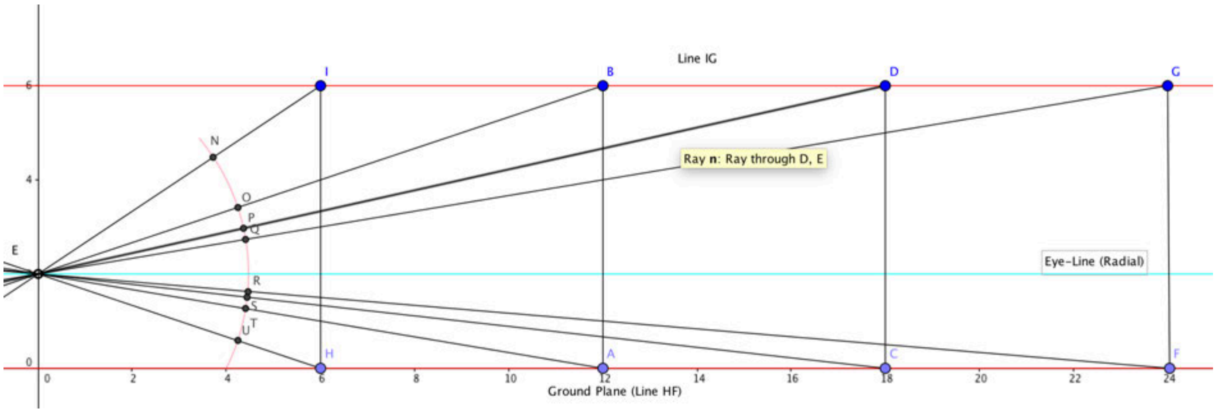
The further any parallel line is from the eye line, the more sudden will be its approach, seen by comparing the size decrease of the angles above and below the eye line.

We can distinguish the visual angles better by drawing an orange circular arc (from U to T) around the eye E, with E at the centre and at any distance (radius) between the eye at E, and the object, line HI. We show where the visual rays from HI, AB, CD, FG intersect with this circular arc at the points N, O, P, Q, R, S, T and U.



The horizontal space IB, on the line IG, appears to advance from N to O on the arc UN. An equal space HA, on the ground plane line HF, appears to move from U to T on the arc UN.

The arc length UT is less than the arc length NO and the arc lengths are in proportion as ΔHEA is to ΔIEB. So, UT is to NO as ΔHEA is to ΔIEB. Similarly, the space BD appears to advance from O to P on the arc UN and an equal space AC appears to advance from T to S on the arc UN.



This continues with DG appearing to move from P to Q on the arc and an equal interval CF appearing to move from S to R on the arc. The angles are in the proportion of their respective arc lengths.

So a horizontal line, plane or continued level surface (e.g. water), if below the eye, as is the ground plane (HF) above, will gradually appear to ascend to the eye line. Similarly, a line or plane above the eye will appear to descend to the eye line. If produced indefinitely, they would both appear to meet on a straight line that is level with the eye.

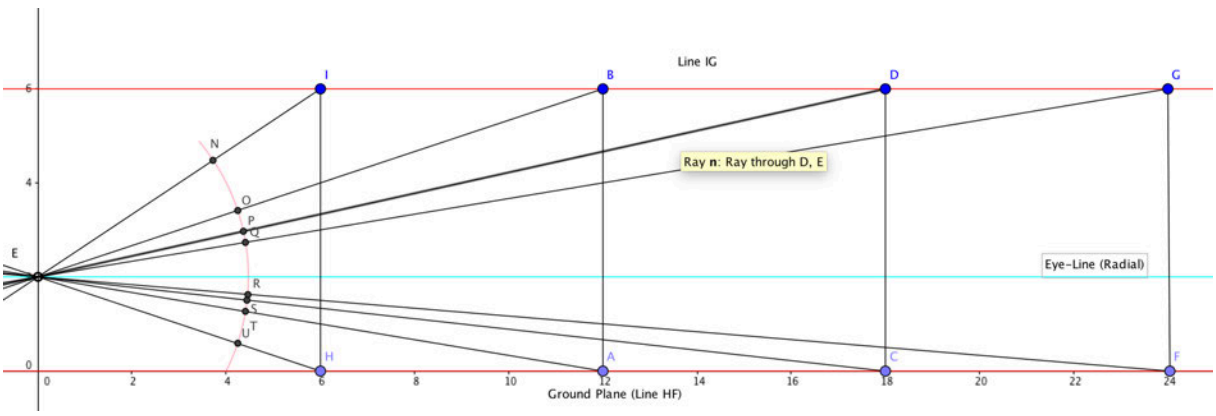
This is the same irrespective of how the plane/line (IG and HF) is situated. Whether horizontal, vertical or inclined, it is always the same. A plane is a plane in all positions, and has no properties peculiar to any position with respect of the eye line.

Every line/plane (such as IG or HF) in which the eye is NOT situate, will appear to approach towards, and at length appear to meet another line/plane which passes through the eye (the cyan eye-line radial), and is parallel to it.

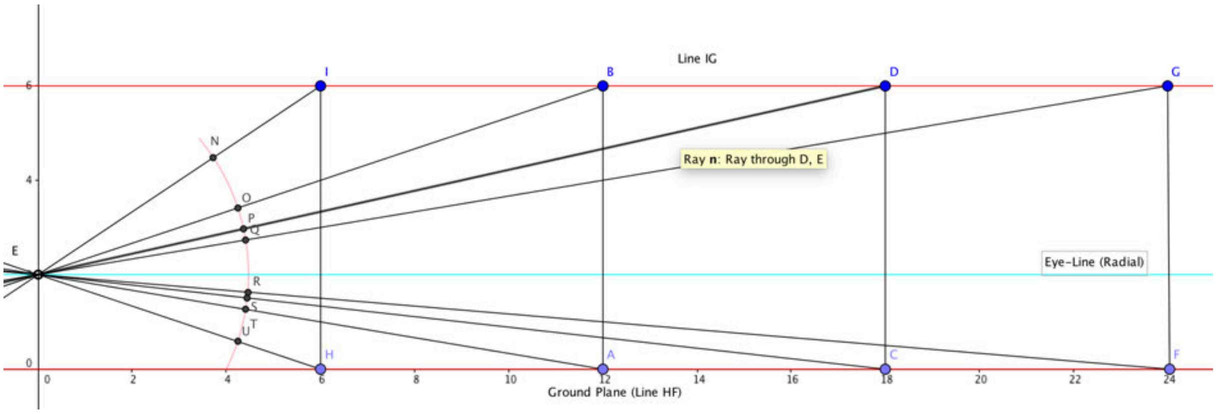
QED.

4.8. What about an object moving into the distance?

Instead of multiple objects in the diagram, let's think of one object, HI, moving farther away to new positions, AB, CD, FG.



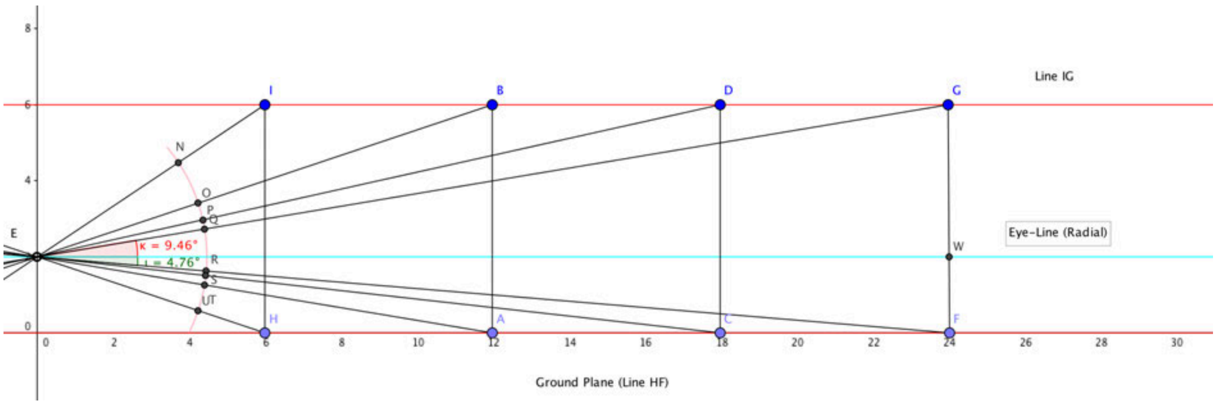
Then the top extreme, the point I of the object HI, will sink lower and lower in the visual field and approach the eye line. At the same time the lower extreme H, will rise higher and higher.



The visual angle of the bottom extreme with the eye line, being closer to the eye line, will always be smaller than that of the upper extreme, irrespective of the distance in the forward direction.

It is smaller to begin with, and will remain smaller, even though both angles approach the same value if the distance is immensely great. The smaller angle will always reach the limit of vision before the bigger one does.

For example, if the visual limit were 5 degrees (which it isn't), then the bottom part of an object *visibly divided* into two parts in the ratio 2:1 from top to bottom, would no longer be distinctly visible at 4.76 degrees. The top part of the object at 9.46 degrees would still be distinct.



At the visual limit an object will start to blur and its intervals, divisions or parts will no longer be distinct. An object will generally disappear bottom first since our eye is usually not far from the ground, and divided parts or intervals closer to the eye line, whose angles are smaller, will reach the limit sooner.

Which part or division of the visual figure becomes indistinct or vanishes first, depends on the figure of the object, the ratios of its divisions and the situation of the eye (its position in 3D space with respect to the object).

With increasing distance, the object and its parts or figure, will continue becoming more indistinct. When the angle between the top extreme and the eye line also falls below the limit of vision, then the whole object disappears from sight.

4.9. What are the limits of vision?

The limits of vision with or without optical instrument extensions vary according to various parameters. The figure, the light illumination, the colour, the atmosphere (medium) are just some of the influencing factors.

A typical average (of the limit of vision) is said to be about 1 minute of space, 1/60th of a degree. This one minute of arc corresponds to a 0.0167 degree planar angle.

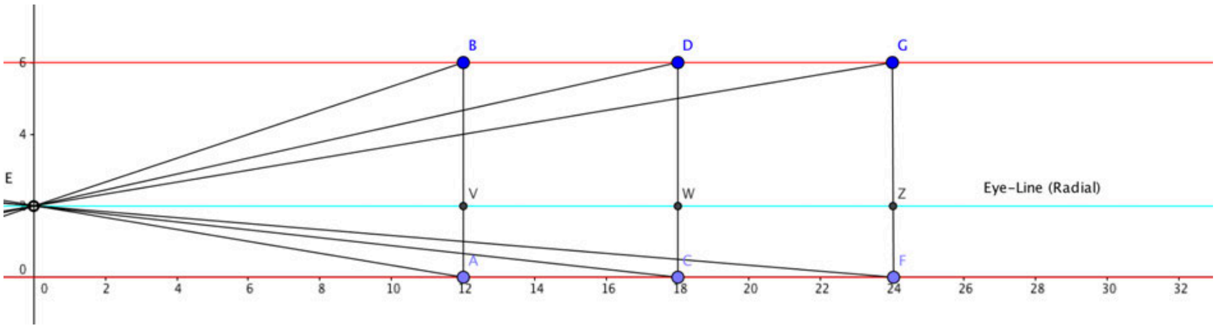
It is estimated that an object in the distance can only be seen up to a distance of about 3500 times its own diameter. So a 1 cm diameter coin, can no longer be seen beyond a distance of 35 metres.

This ratio depends also on the position of the eye, and the viewing angle subtended at the eye. It depends on how high and how much “off centre” it is, with respect to a direct line of sight to the centre of the object. This eye line is parallel to the ground plane.

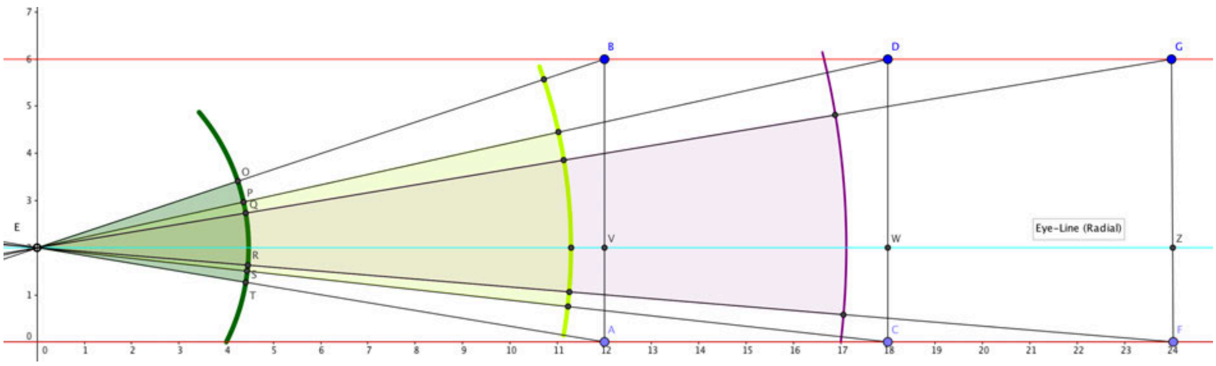
5. More consequences

5.1. Appearances and representations

Let AB, CD and FG be three vertical parallel objects of equal height, at the respective distances EV, EW and EZ from the eye, at E. Then those objects appear in the proportion of the angles ΔAEB, ΔCED and ΔFEG.



Again we show the arcs of a circle, whose centre is E. Any concentric circle of centre E between the eye and object can be taken. The angles will be the same. The portions of the arcs, intercepted between the visual rays, are the measures of each angle respectively, and determine the apparent proportions of the objects AB (dark green arc/sector), CD (light green arc/sector) and FG (purple arc/sector).



Whatever number of degrees their arcs contain, the angles which they subtend, are in the same ratio. This define the appearance of the object (or of its individual visible divisions).

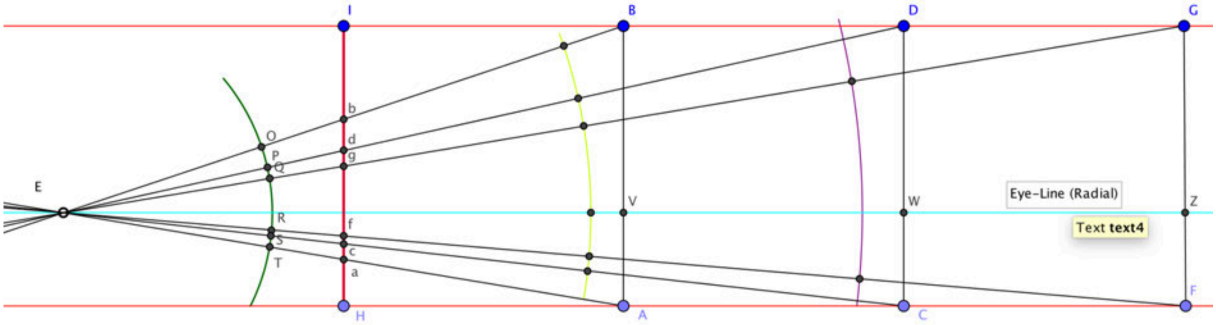
However, in <u>perspective drawing</u>, the <u>representation</u> of the object, its image or picture, is <u>always</u> on a <u>plane</u> and not on an arc or sphere surface.

Let's look closer at what that means by analysing how the apparent magnitudes of objects are determined.

5.2. Apparent Magnitudes of objects

Let HI be a section of a Plane (dark red), in a direct position, perpendicular to the eye line and its plane (cyan blue) passing through the eye. Let AB, CD, FG be objects of equal size.

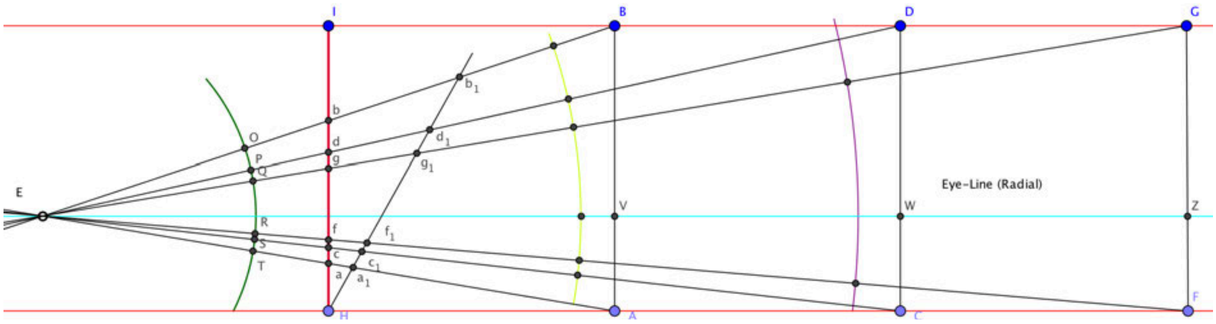
The eye, at E, is NOT equidistant from the extremes of the objects. The eye line is closer to A than to B, closer to C than to D, and closer to F than to G.



The intervals or spaces ab, cd and fg, between the points where the visual rays cut and pass through the red HI plane, create a picture. These intervals or spaces are the proportions only of the representations of the objects AB, CD and FG when projected onto the plane.

These spaces are NOT their APPARENT magnitudes, which is the size of the angle subtended at the eye. Their apparent magnitudes are the portions, labelled TO, SP and RQ, of the dark green arc, labelled TSRQPO, of which the eye at E, is the centre.

This is a very important distinction and is often confused and/or obfuscated. To see the difference, lets introduce a new plane Hb_1 inclined to the eye line plane



If the picture is inclined to the eye line plane, as in Hb_1, with the objects and the eye staying as they were, then the Representations a_1b_1, c_1d_1 and f_1g_1 on that Plane, are very different from the proportions of ab, cd, and fg on the original red plane HI.

However, the APPARENT proportions of both are the same, namely the portions on the arc TSRQPO, which are the measures of the angles AEB, CED and FEG.

These portions of arcs are the real APPARENT magnitudes of the objects and remain the same irrespective of whichever plane is used to represent the image of the object.

If this is doubted, notice that the objects AB, CD and FG have not moved or changed with respect to each other and likewise the eye, at E has not varied its position. Consequently, the

angles AEB, CED and FEG have not been altered. The objects AB, CD and FG and the angles they subtend must necessarily appear the same when represented on any plane surface, in any position, which is cutting the visual rays EA, EB, EC, ED, EF and EG.

It is important to remember this difference in the nature of vision and what we see. This is the correct way to determine the apparent magnitudes of objects. It is the angle subtended at the eye that is important and not the magnitude of an image representation on a plane.

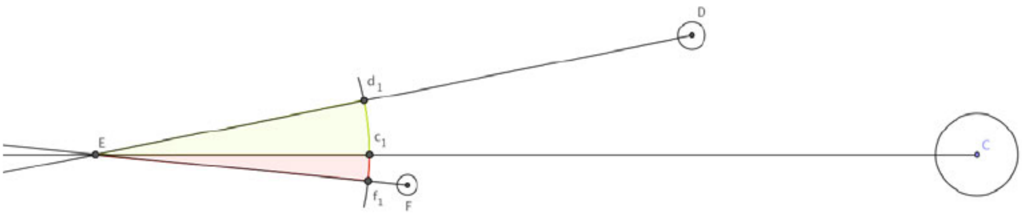
Multiple different representations of magnitudes can be created on a plane (e.g. line b_1d_1), by varying the inclination of the plane, or by viewing obliquely. None of the lines drawn on the picture plane, when measured, would be correct.

5.3. Apparent distances, or Bearings

The apparent distances, or bearings of objects with respect to each other, are determined in a similar way to the magnitudes of objects. It must be done by the degrees on the arc of a circle (sphere), which measure the angles that the objects make with the eye.

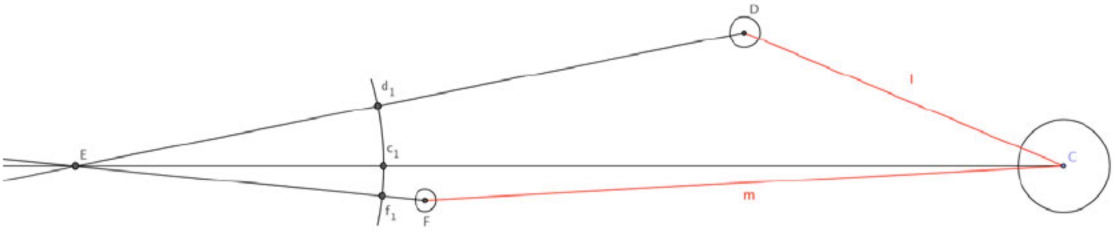
Let C, D, and F be three round objects of different sizes and distances from E, the eye.

The three visual rays CE, DE, FE create two angles DEC and CEF, that are the optic angles of their apparent distances, or bearings at the station E, the eye. These are the angles subtended at the eye by the pink and green areas shown below and are always correctly determined by the arcs c_1f_1 and c_1d_1.



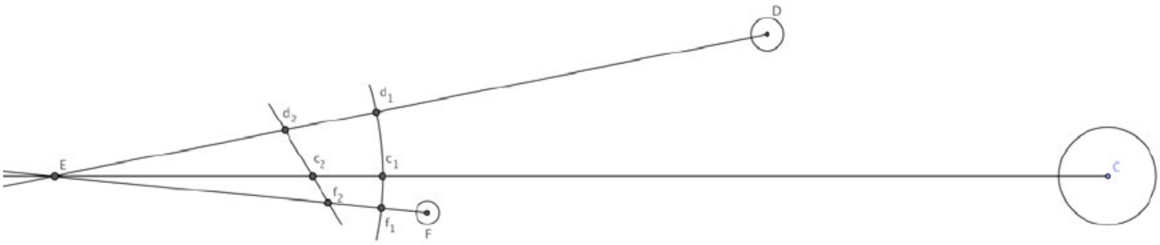
The arc angles are the TRUE APPEARANCE of objects, or sometimes it is said that the eye is in the TRUE POINT of VIEW.

Now in reality the distance from C to F (red line m) is nearly double that of the distance from C to D (red line l), yet the angle CEF, is much less than the angle DEC, as can easily be seen by the coloured arc segments c_1f_1 and c_1d_1 on the Arc d_1f_1.



So the angles tell us nothing about the real distances between objects. C and D are in reality much closer to each other than C and F are, but the angle that C and D make is seen to be much bigger than the angle made by C and F.

Suppose also that f_2d_2 is a section of a plane at any inclination, between the eye and the objects. Then the appearances, or places of the objects on this plane, are at f_2, c_2 and d_2.



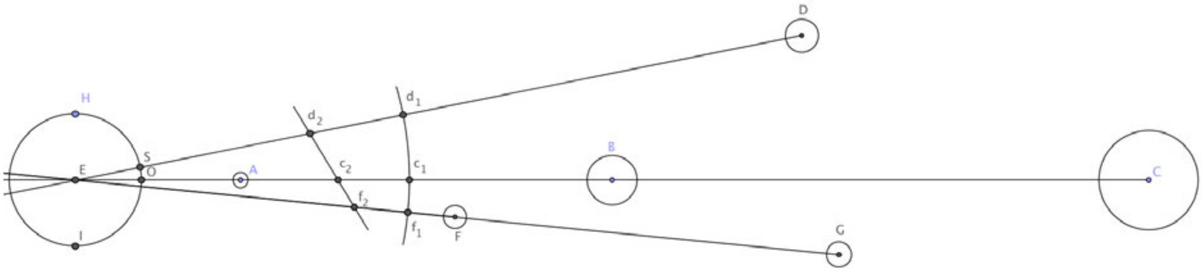
Representation on a plane does not change our inability to distinguish or determine real distances and sizes.

5.4. How do objects relate to each other in distance and bearing?

Let's imagine that these objects C, D, and F are stars at an immoderate distance from each other and from the eye, at E.

Stars are not seen during the day. They are only seen on a dark background as white coloured discs. What is presented to our eye, has no 3rd dimension as seen above. No depth cues are available from colour, texture, shading etc. We are only witnessing the geometric figure, form and outline of an object. It is not possible to ascertain their real distances or situations because the visual angles tell us nothing about the real distances or real positions.

If the star C was at B or anywhere else on the direction EC, and if the star F was at G, or anywhere else in the direction EG, their apparent places are still at c_1 and f_1, on an imaginary arc of the sky. This is the arc of a circle, centred at E, and equally distant from the eye at all points.

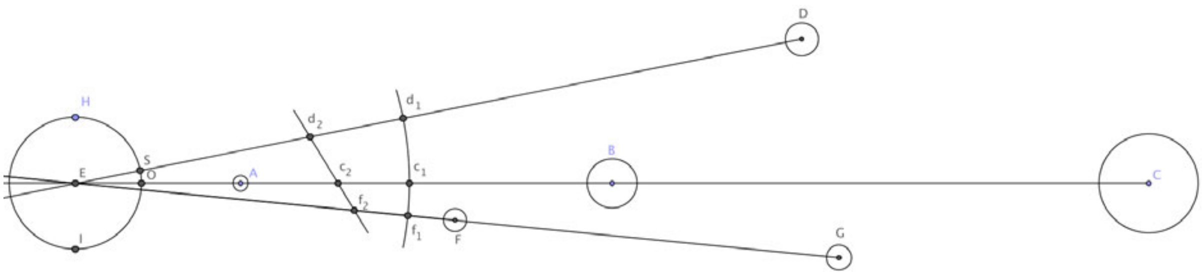


Remote objects (e.g. stars) appear equally distant from the eye. They appear as if they were all in the same concave sphere, of which the eye is the centre. The sphere of the sky, d_1f_1 is <u>equally distant</u> in every part from the eye.

So is the star C, really at B or really at A? Well no sensible difference in bearings is seen, even over a whole supposed Earth orbit diameter. If we assume heliocentricity, then a difference should be seen after 6 months, but there never is.

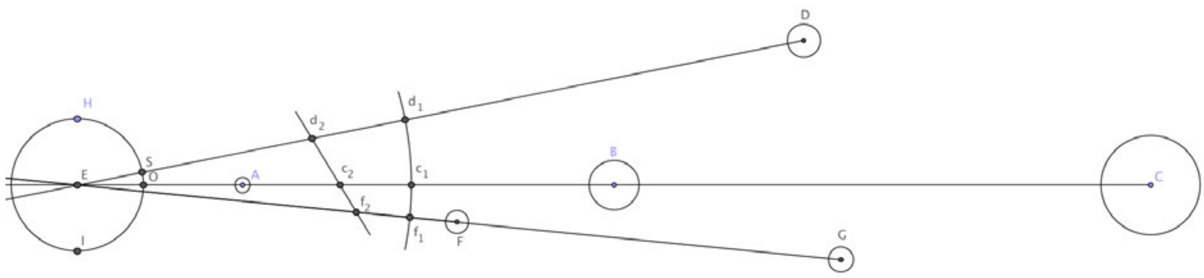
Their apparent places on the arc d_1f_1, does not give us any information about their real distances. The portions f_1c_1, c_1d_1 of the arc d_1f_1 are the measures only of their apparent distances <u>with respect to each other</u>. These apparent distances are the angles d_1Ec_1, c_1Ef_1 or their original angles DEC and CEF.

This is also known as the **Celestial sphere**. This sphere has its centre at E and boundary at the arc d_1f_1 or any equivalent concentric circle of any radius between the eye and the object such as the circle HI around E. It can be a bigger or smaller (radius) sphere, since the <u>relative angles</u> and <u>proportions</u> of the <u>respective arc lengths</u> will always remain the same.



From the centre E, which is also the centre of the arc d_1f_1, the star C will appear on the arc sphere HI at O, and the star D will appear at S. They are in the same position, situation and distance in proportion to the radius EO to Ec_1 (concentric circles) as on the imaginary arc d_1f_1.

So whether the star is at A, B, C or anywhere in the direct line EC, its apparent or representative place on the Celestial sphere HI, will still be at O.



There is then no difference in the position of the star seen, whether it be A, B or C, but the apparent magnitude or size at O, will nearly* be in proportion to its distance. (*Tangent lines to an assumed spherical body cannot measure a perfect diameter because of the viewing angle).

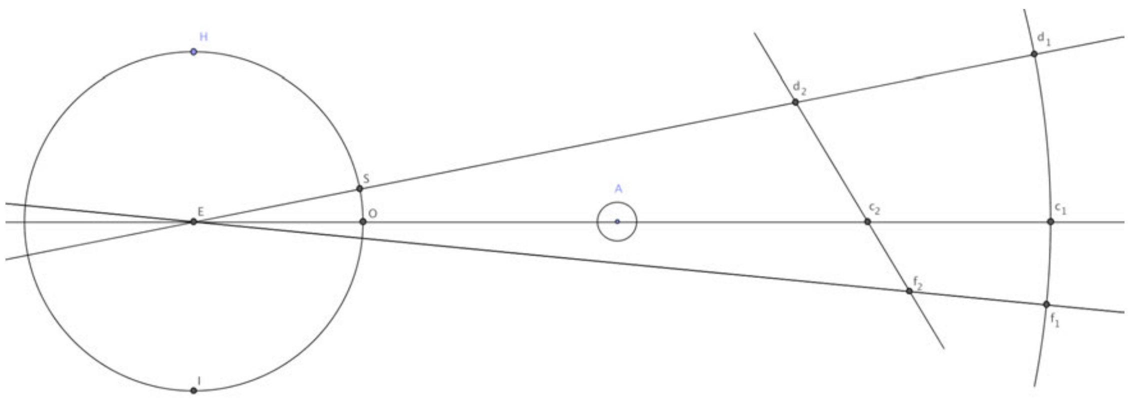
It is now hopefully clear that there is a manifest difference between the representation of an object on a plane and its true appearance, the visual angle subtended at the eye.

This difference is greater when the eye is nearer to the object. By nearer, we mean the distance of the object such as F or D, from the eye line EC, (horizontally or vertically depending on the visual field of view) because the visual rays must necessarily cut any plane on which they fall more and more oblique, the further the intersections f_2 and d_2 are from that point c_2, where a perpendicular line from the eye would cut the plane, along the eye line to the centre of the object.

So the representations of objects on a plane, cannot be in proportion to their true appearance, but must deviate continually more and more, as they fall farther from the point c_2, as described above.

When representing objects on a plane, we can never maintain the same proportions as their true appearance when the objects are in different positions that are farther or closer to the eye line.

The arc length c_1f_1 is nearly the same as the line segment c_2f_2, since f_1 and f_2 are close to the eye line. The arc length c_1d_1 and the line segment c_2d_2 are already noticeably unequal since d_1 and d_2 are farther from the eye line Ec_1



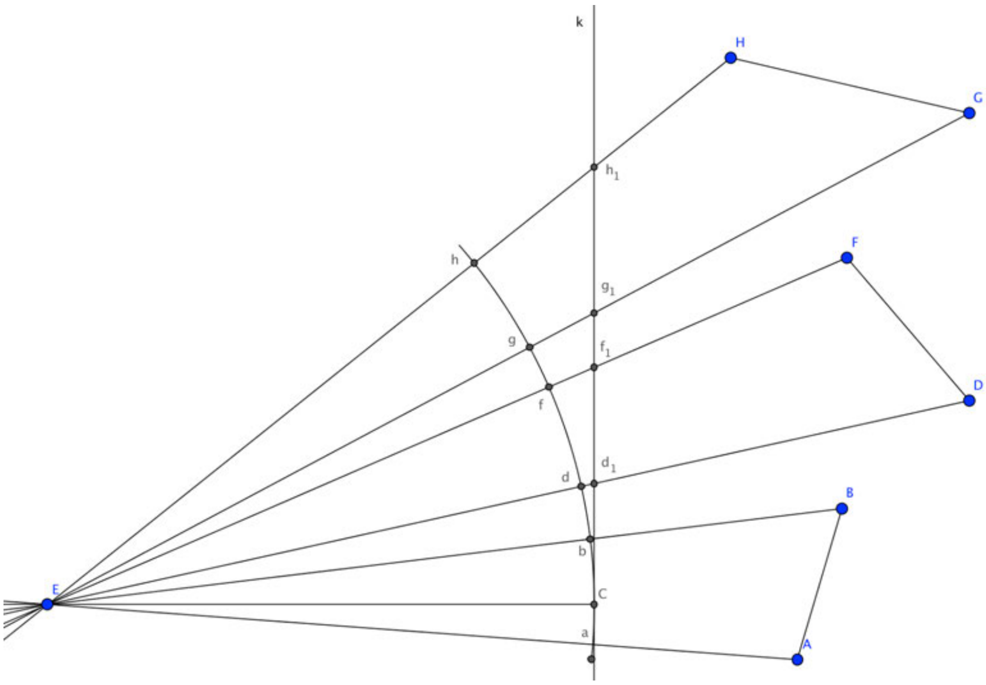
Now in contrast to this, a circle (or spherical surface) has every point equally distant from its centre. Every straight line in every direction that is directed to its centre E, is therefore perpendicular to its surface (e.g. at d_1, c_1, f_1, S, O, etc.) and none of these lines will cut the arc circle obliquely.

<u>So the true representations and true appearance of objects, can only be correctly depictured on figures and surfaces where visual rays intersect perpendicular to the surface, such as a circle or a sphere</u>.

Remember: The representation of objects on the concave surface of the retina are equivalent to their representation on the concave surface of any concentric sphere situated outside the eye but between it and the object. This is because the angles subtended are in the same proportions as the arc segment lengths.

When viewed with the eye at the centre, the true appearance of the object, is the same in both cases.

To show this clearer, let AB, DF and GH be three objects situated as shown below with respect to the eye, at E. The visual rays EA, EB, ED, EF, EG and EH are drawn from the extremes of the objects through the eye at E. If the eye remains at E, then the objects will <u>appear</u> the <u>same</u> on <u>any</u> <u>surface</u> that <u>cuts</u> the visual rays EA, EB, etc.



Let ak be a plane section intercepting the rays. The object AB has its representation a_1b_1 on the plane nearly equal to ab on the arc, whereas d_1f_1 is obviously larger than df on the arc. The interval g_1h_1 is even larger than gh on the curve, because it is more remote from the point C where the eye line meets the plane perpendicularly.

The <u>true</u> <u>appearance</u> is gh on the arc curve or spherical surface, for they <u>both represent the same object, from the same point of view, which CANNOT vary in its appearance</u>.

Therefore, the representations of objects on various surfaces, in various situations, although they can differ greatly in figure, will, if correctly represented, appear the same only in the true point of view of the eye.

Note: Many artists (and illusions) require the use of a peephole, where the eye is fixed in a specific position, so that the perspective will work as intended by the artist. As an example, think of viewing a circle obliquely. It is usually an ellipse, but an artist can stage the scene so that the circle is seen circular when viewed through a peep hole. If we know the position where the eye will be fixed when the image is viewed, then objects can be positioned so as to be seen as intended. A good example of this is the Ames room "illusion".

The three objects AB, DF and GH though of various dimensions appear to be equal. This is not due to their distances, but due to their positions with respect to the eye.

They are being seen under the equal angles, AEB, DEF, GEH, as the segments ab, df, gh of the arc abdfgh clearly show. However, their representations on the plane ak, are unequal. Those nearer the centre C such as a_1b_1 are the least, whereas d_1f_1 is greater than ab, and g_1h_1 is even greater than d_1f_1.

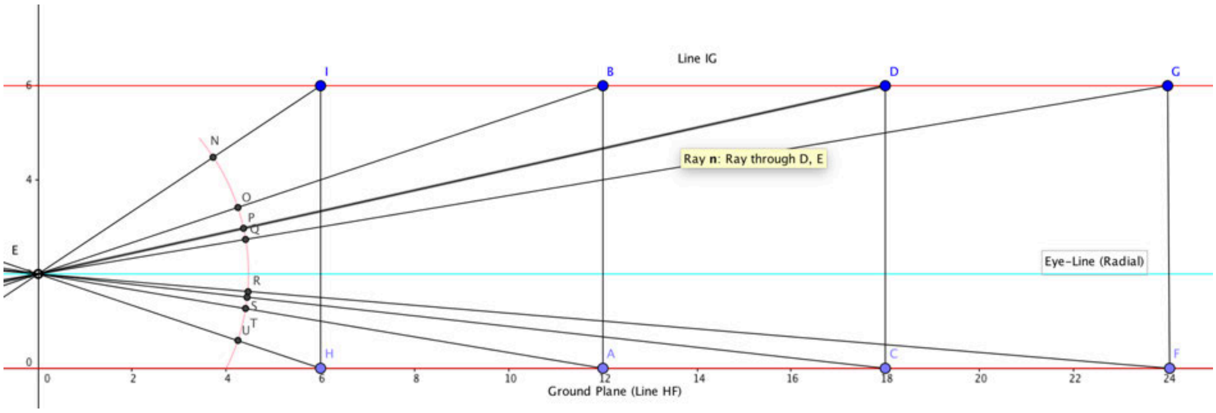
The representations a_1b_1, d_1f_1 and g_1h_1 on the plane have the same appearance to the eye at E, as their corresponding Images ab, df and gh on the spherical surface. The eye is the true point of view at E, and so they are seen by the same visual rays as the original objects themselves and consequently under the same optic angles. From Theorem 1 their appearances are then the same on either surface.

To summarize this section

A representation on any plane, situated between the eye and the object, subtend the same angle with the eye as do the images of the object when represented on a spherical surface.

This spherical surface is the curved retina at the back of the eye, with E being the centre of the globular eye. Or equivalently it can be any concentric sphere, or circular arc in a cross section side view, that lies between the eye and the object.

6. Notes on DV Geometrical diagrams.



These diagrams can be interpreted in two ways.

As a Side View. We have sliced the head of the person, with a vertical plane, down the middle of the eye, and are looking at the cross section created. In the drawing, E and the orange arc UN have the same visual angle ratios, as the arc formed on the retina by the visual rays. We can draw the visual rays and determine the angles, that must be entering the eye from objects oriented vertically in the Y Axis. The eye is getting rays from above and below its own eye line. The eye of the person is looking up and down.

As a Top View. We have sliced the head of the person, with a horizontal plane, across the middle of the eye, and are looking at the cross section created. We can draw the visual rays and determine the angles, that must be entering the eye from objects oriented horizontally in the X Axis. The eye is getting rays from left and right in this diagram. We can rotate the diagram to the left 90 degrees to visualize this better, but that doesn't change the rational geometry.

The side and top view are equivalent. The side view is showing the visual rays of the Y Axis (up and down), whilst the top view is showing the visual rays of the X Axis (left and right view) from the point of view of the eye.

The two dimensional propositions we have discussed apply to both. The horizontal (latitude) and vertical (longitude) have equivalent visualizations, because the eye is spherical and the geometry of vision is bipolar elliptic.